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Stochastic optimal control, international finance and debt

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This paper is dedicated to the memory of James Tobin (1918–2002), teacher and friend

Abstract

We use stochastic optimal control-dynamic programming (DP) to derive the optimal debt/net worth, consumption/net worth, current account/net worth, and endogenous growth rate in an economy – which could be a country, region or sector within a country. Unlike the literature that uses an intertemporal budget constraint or the Maximum Principle, the DP approach does not require perfect foresight or certainty equivalence. Our results are generalizations of the Merton model, and are explained graphically within a mean–variance context. Two examples are provided to illustrate the usefulness of our technique in predicting debt crises.

JEL classification: C61; D81; D9; F34
Keywords: Stochastic optimal control; Debt; International finance; Vulnerability to external shocks; Sustainable current account deficits

1. Different approaches to intertemporal optimization in open economies

Several noteworthy debt crises have occurred in recent years. In the case of South East Asia in 1997, data on the credit rating of bonds issued in the first half of the 1990s suggest that investors in emerging market securities paid little attention to credit risk, or that they were comfortable with the high level of credit risk that they

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were incurring. [1] The compression of the interest rate yield spread prior to and the subsequent turmoil in emerging markets have raised doubts about the ability of investors to appropriately assess and price risk. In the US agricultural debt crisis case, the boom of the 1970s was stimulated by a substantial rise in crop prices resulting from inflation, the growth of export demand and availability of credit. A dramatic rise occurred in the ratio of debt/value added. In the fall of 1979, the Federal Reserve Board tightened its monetary policy to reduce inflation and interest rates soared. Moreover, the resulting appreciation of the US dollar exacerbated the decline in foreign demand for US agricultural exports. The net effect was that many farmers found that they were not able to service their debts. In the first half of the 1980s, bankruptcies, defaults and bank failures resulted. In each case, a benchmark is needed to evaluate to what extent a debt deviates from its optimal value.

In our model economies borrow to finance investment and growth as well as consumption. Debt to finance capital formation involves two risks. One is the return on domestic investment. The second is the variable interest rate on debt. A *benchmark* is provided to evaluate to what extent the debt deviates from its optimal value. The variables of interest discussed in this paper are the *optimal debt*, *current account*, *growth rate*, and *consumption*. Our technique is applicable to any open economy – which could be a country, region or sector within a country. In the concluding section, we provide two examples of the use of the technique. One concerns the debt crisis in emerging markets and the second, the US agricultural debt crisis.

Optimality conditions should satisfy several criteria: (a) they involve observable and measurable variables, (b) if followed, would maximize the value of sensible criteria and (c) do not produce very bad results if there is imperfect knowledge or errors of measurement. Several approaches have been used to derive optimality conditions in open economies. The dominant ones use either "an intertemporal budget constraint" (IBC) or the Maximum Principle of Pontryagin. It is recognized that these approaches are deficient [2] in satisfying criteria (a)–(c) above.

As a rule, economists have used the Maximum Principle of Pontryagin to derive optimal control laws. This is an "open loop" type of optimization method that yields an entire sequence of controls to be followed from initial conditions. Half of the initial conditions must be obtained from transversality conditions which imply the solution of differential equations. Given the likelihood of unpredictable disturbances, errors of measurement, formulation and implementation, the overall system will not be stable unless converted into a feedback form. This is to be expected since the optimal path to the desired target is unique. It is clearly advantageous in economics to derive policies in feedback form, where the next move depends upon the current state, since these types of policies are self-correcting and robust to perturbations. [3]

[1] In the Asian crises, spreads hardly increased in the months prior to the floatation of the Bhat. The credit rating agencies and the market analysts all failed to signal the Asian crises in advance. They downgraded these economies only after the crises. See International Monetary Fund (1997, 1998, 1999a,b).

[2] See Gandolfo (2001, ch. 1 and 18), Hahn and Solow (1995), and Stein and Paladino (1997).

[3] This was the contribution of Infante and Stein (1973).

When the economic system is deterministic, the controller can predict the future state of the system knowing the initial conditions and the controls used in the past. In a stochastic system – such as our case where both the productivity of investment and the interest rate are stochastic and hence unpredictable – the controller cannot predict the future, because there are many paths that the system states may follow given the initial conditions and the past controls. *Since the controller cannot predict the future, the Dynamic Programming (**DP**) approach is used, where the optimal controls are based upon the observed state.*

The paper is divided into several parts. The text explains the economic significance and intuition behind our results, and the mathematical derivations are in the appendix, available upon request. An appealing feature is that we can explain the DP results in terms of the mean–variance analysis and a corresponding simple diagram.

In part 2, we describe the endogenous growth model of an open economy subject to productivity and interest rate shocks. This growth model is related to models used in the literature. Box 1 summarizes the basic equations. Part 3 sets up the stochastic optimal control/dynamic programming approach and states the results in Box 2 and as Propositions I, II, III, IV, V. They provide us with the appropriate *benchmarks* that satisfy criteria (a)–(c) above. Part 4 shows that these results have very simple and clear relations to the mean–variance approaches developed by James Tobin to whom this paper is dedicated. [4] Our work is shown in part 5 to be a generalization of Merton's model of portfolio selection to an open economy. Since both our papers use DP, our results have comparable forms; and both are very different from the literature that uses the IBC or the Maximum Principle. Part 6 derives the optimal expected growth rate, and part 7 derives the optimal expected current account/net worth. In concluding part 8 we cite examples of how our analysis could have provided warning signals of several debt crises.

2. A continuous time infinite horizon model

The endogenous growth model summarized in Box 1 is a generalization of the models in the literature. There are two sources of uncertainty: the return on capital, and the interest rate on loans. It is important and realistic to stress that there is a correlation of these two sources of uncertainty, which differs among economies. The model is in real terms and is formulated in terms of the stochastic calculus. To formulate a stochastic control problem associated with the model, we must specify the *state* and *control* variables, the *constraints*, the *dynamics* of the state process and the *criterion* to be optimized.

There are many criteria of optimality. We use the standard criterion, the maximization over an infinite horizon of the expectation (E) of the discounted ($\delta > 0$) value

[4] James Tobin (1918–2002) developed the content of his "Liquidity Preference" RES 1958 paper in 1950, in his graduate course on macroeconomics where Stein was a graduate student. It is fitting that this paper be dedicated to this "gentleman and scholar".

of the utility of consumption $U(C(t))$. This is the right hand side of Eq. (1). The utility function (1b) or (1a) and set Γ of constraints and controls are discussed below.

In the international finance interpretation of our model, consumption, $C(t)$ in Eq. (2), is GNP less investment plus net capital inflow, which is current net external borrowing. The GNP is the GDP less net interest payments on the external debt. The components of Box 1 are now discussed.

Box 1. Equations of the stochastic growth model

$$V(X) = \max_{\Gamma} E\left\{ \int_0^\infty U(C(t))e^{-\delta t}\,\mathrm{d}t \right\}, \tag{1}$$

$$U(t) = \ln C(t), \quad \gamma = 0, \tag{1a}$$

$$U(t) = (1/\gamma)C^{\gamma}(t), \quad \gamma < 1, \tag{1b}$$

$$C(t)\,\mathrm{d}t = Y(t)\,\mathrm{d}t - r(t)L(t)\,\mathrm{d}t - I(t)\,\mathrm{d}t + \mathrm{d}L(t) > 0, \tag{2}$$

$$Y(t) = [b\,\mathrm{d}t + \sigma_2\,\mathrm{d}w_2]K(t) = b(t)K(t), \quad \mathrm{d}w_2 = \varepsilon_2\sqrt{\mathrm{d}t}, \\ \varepsilon_2 \sim N(0,1)\mathrm{iid}, \tag{3}$$

$$E[Y(t)/K(t)] = b\,\mathrm{d}t, \tag{3a}$$

$$\mathrm{var}[Y(t)/K(t)] = \sigma_2^2\,\mathrm{d}t, \tag{3b}$$

$$\mathrm{d}K(t) = I(t)\,\mathrm{d}t, \tag{4}$$

$$\mathrm{d}L(t) = r(t)L(t)\,\mathrm{d}t + [C(t) + I(t) - Y(t)]\,\mathrm{d}t, \tag{5}$$

$$r(t)L(t)\,\mathrm{d}t = rL(t)\,\mathrm{d}t + \sigma_1 L(t)\,\mathrm{d}w_1, \quad \mathrm{d}w_1 = \varepsilon_1\sqrt{\mathrm{d}t}, \quad \varepsilon_1 \sim N(0,1)\mathrm{iid}, \tag{6}$$

$$E[r(t)L(t)\,\mathrm{d}t] = rL(t)\,\mathrm{d}t, \tag{6a}$$

$$\mathrm{var}[r(t)L(t)\,\mathrm{d}t] = E[r(t)L(t)\,\mathrm{d}t - rL(t)\,\mathrm{d}t]^2 = E[\sigma_1 L(t)\,\mathrm{d}w_1]^2 \\ = (\sigma_1 L(t))^2\,\mathrm{d}t, \tag{6b}$$

$$E[\varepsilon_1\varepsilon_2] = \rho, \quad 1 \geqslant \rho \geqslant -1, \tag{7}$$

$$X(t) = K(t) - L(t), \tag{8}$$

$C(t)$ = consumption, $Y(t)$ = GDP, $L(t)$ = debt, $I(t)$ = investment, capital = $K(t)$, $r(t)$ = rate of interest, $X(t)$ = net worth = capital − debt = $K(t) - L(t)$; Brownian motion, w_1, w_2. Constraints: $\Gamma = [C(t) > 0, X(t) > 0]$.

2.1. Production function: Uncertainty concerning the return on capital

The production function (3) states that the GDP or value added $Y(t)$ is proportional to capital $K(t)$. The ratio of real output/capital $Y(t)/K(t) = b(t)$ is the return on capital. The deterministic part is the mean return b, with no time index, and the stochastic part involves Brownian motion term $\sigma_2\,\mathrm{d}w_2$, whose the mean is zero and the variance is $\sigma_2^2\,\mathrm{d}t$. The stochastic term arises from variations in the prices of output and of inputs – terms of trade – as well as from the physical productivity due, for example, to the weather, floods, or disease. The stochastic term may exhibit great variations over short periods of time. The change in capital $\mathrm{d}K(t)$ in Eq. (4) is the investment over the period $I(t)\,\mathrm{d}t$.

2.2. Debt payments uncertainty

In Fleming and Stein (2001), we considered a discrete time-finite horizon model where borrowing is in the form of short term debt, which must be repaid with interest at maturity. Here, we assume that there is no maturity but the debt must be serviced continually at a *variable* real interest rate $r(t)$.

The change in the debt equation (5) is the current account deficit. It is the sum of the interest payments on the debt $L(t)$ at interest rate $r(t)$, plus the trade deficit equal to $C(t) + I(t) - Y(t)$ the sum of consumption plus investment less GDP.

The real interest payments $r(t)L(t)$ in Eq. (5) are stochastic. Eqs. (6), (6a), (6b) describe the probability distribution function of the stochastic service payments on the debt. The interest costs on the debt $r(t)L(t)\,\mathrm{d}t$ are distributed normally with a mean of $rL(t)\,\mathrm{d}t$, in Eq. (6a). Thus r is the mean or expected real rate of interest. The variance is described in (6b) equal to $E[r(t)L(t)\,\mathrm{d}t - rL(t)\,\mathrm{d}t]^2 = E(\sigma_1 L(t)\,\mathrm{d}w_1)^2 = \sigma_1^2 L(t)^2\,\mathrm{d}t$. These two moments are implied by Eq. (6).

2.3. The correlation of the shocks to growth and to the interest rate

Eqs. (3), (3b) and (6), (6b) describe the uncertainty. The two stochastic terms $\mathrm{d}w_1$, $\mathrm{d}w_2$ in Eqs. (3) and (6) are interrelated. The first concerns the variability of the real rate of interest, Eq. (6b), and the second concerns the variability of the return on capital, Eq. (3b). We consider the general case, Eq. (7), where the two shocks are not necessarily independent: $E(\mathrm{d}w_1\mathrm{d}w_2) = E(\varepsilon_1\varepsilon_2)\,\mathrm{d}t = \rho\,\mathrm{d}t$. Correlation coefficient ρ could be positive, zero or negative, which varies among economies, regions and sectors and over time for each one.

Often in the macroeconomy, a rise in the return on capital stimulates an economic expansion, which leads to a rise in interest rates – a positive correlation. A very different situation exists when there has been a change in monetary policy or a financial crisis. [5] For example, in the mid 1970s the agricultural economy expanded driven by a

[5] See Friedman and Schwartz (1963, p. 312). The negative correlation between growth and the yield on lower grade bonds is crucial in understanding the severity of the great depression and financial crises.

rise in export demand. In 1979 there was a switch in monetary policy that raised interest rates. The latter led to an appreciation of the US dollar and adversely affected export demand and return on capital. In that case, the correlation was negative.

3. The dynamic programming solution

In this section we state the dynamic programming solution, which is derived in the mathematical appendix, available upon request. Then it is given an economic interpretation by showing how it is related to a mean–variance model and is a generalization of the Merton's model. Thereby we are able to demonstrate almost all of our results using a mean–variance technique and a simple graph.

The state variable is net worth $X(t)$ defined in Eq. (8). It is capital less debt. Capital [6] is $K(t)$ and its change is Eq. (4). The dynamics of the state variable net worth $X(t)$ are expressed in Eqs. (9)–(11). The change in net worth $\mathrm{d}X(t)$ is Eq. (9).

$$\mathrm{d}X(t) = \mathrm{d}K(t) - \mathrm{d}L(t). \tag{9}$$

Substitute $\mathrm{d}K(t)$ from Eq. (4), and the change in the debt from Eqs. (5) and (6) to obtain Eq. (10).

$$\mathrm{d}X(t) = [bX(t) + (b-r)L(t) - C(t)]\,\mathrm{d}t - L(t)\sigma_1\,\mathrm{d}w_1 + (X(t)+L(t))\sigma_2\,\mathrm{d}w_2. \tag{10}$$

The object is to maximize the expected present value of utility equation (1). The choice of utility function is very important. Assume that utility is HARA, Eq. (1b) for $\gamma < 1$, or Eq. (1a) when $\gamma = 0$. Risk aversion is positive, $(1-\gamma) = -\mathrm{d}\ln U'(C)/\mathrm{d}\ln C > 0$. Eq. (1) becomes Eq. (1c).

$$V(X) = \max_{\Gamma} E\left\{\int_0^\infty (1/\gamma)C(t)^\gamma \mathrm{e}^{-\delta t}\,\mathrm{d}t\right\}, \quad \gamma < 1,\ \ \gamma \neq 0. \tag{1c}$$

There are several advantages to the use of the HARA function. First: it reduces the dimension of the problem and allows us to solve the model analytically. Second: it is scale independent. It is valid regardless of the size of the economy. Mathematically [7] this is expressed by the property $V(X) = (1/\gamma)AX^\gamma$ for a suitable constant $A > 0$. In the logarithmic case $V(X) = A\ln X + B$. The constant A is determined by the DP equation. In the logarithmic case $A = 1/\delta$, the reciprocal of the discount rate. Risk aversion requires that $\gamma < 1$. If we assume that $\gamma \leqslant 0$, we do not have to make any restrictions on the discount factor δ, which would be needed if we only assumed that $\gamma < 1$. Our benchmark system with net worth $X(t) > 0$ is constrained to preclude Ponzi schemes, where debt is refinanced by further borrowing. [8] The HARA utility function allows us to use as controls the *ratios* of: debt/net worth $f = L/X =$

[6] The exact same mathematical results are obtained if we define capital K^* in the Frank Knight sense as the discounted value of current GDP at a discount rate equal to b the mean return, $K^* = Y(t)/b$.

[7] See the mathematical appendix for proofs of statements made in the text.

[8] In a Ponzi scheme net worth becomes negative.

$k-1=$ capital/net worth $k-1$, and consumption/net worth $c=C/X$. Eq. (10a) is in terms of the control ratios f and c. Net worth $X(t)$ cannot become negative, a Ponzi scheme is not possible, because as $X(t)$ declines to zero, so does $\mathrm{d}X(t)$.

$$\mathrm{d}X(t) = [(b-c)+(b-r)f]X(t)\,\mathrm{d}t - fX(t)\sigma_1\,\mathrm{d}w_1 + (1+f)X(t)\sigma_2\,\mathrm{d}w_2. \tag{10a}$$

The optimization (1c) is subject to the dynamic equation (10a) and to the constraints $C(t)>0$, $X(t)>0$. The *control* variables are consumption ratio $c(t)$ and the debt ratio $f(t)$. A simplifying assumption is that the controls $c(t), f(t)$ can be varied instantaneously and costlessly. Given the nature of the uncertainty, the controller cannot anticipate the future. The admissible controls are chosen using any information known up to time t. We therefore consider the controls which enter as feedback functions of the state $X(t)$. This is fundamentally different from the "forward looking/certainty equivalent" models in the economics literature, [9] but it is the same orientation as the Merton approach in mathematical finance.

The equations for the optimal ratio of debt/net worth f^* and consumption/net worth c^* are obtained from DP equation (11), which is derived in the mathematical appendix, available upon request. The crucial *objective* parameters in the maximization are: (i) *controls* $f=$ debt/net worth, $c=$ consumption/net worth, (ii) *parameters* $b=$ mean return on investment, $r=$ mean interest rate, standard deviations of the return σ_2 and interest rate σ_1, ratio $\theta=\sigma_1/\sigma_2$ and $\rho=$ correlation of return and interest rate. The *subjective* parameters are risk aversion $(1-\gamma)$ and discount rate δ. The discount rate does not enter into the maximization with respect to the optimal debt ratio, but it does for the optimal consumption ratio.

In the text we start from the DP equation and show how it can be related in terms of mean–variance analysis. On the basis of simple graphs, one obtains clear and economically significant and sensible results.

Dynamic programming maximization:

$$\begin{aligned}\delta/\gamma &= \max_c[(1/\gamma)c^\gamma/A + (b-c)]\\ &\quad + \max_f\{[(b-r)f] - (1-\gamma)\sigma_2^2/2[(f^2\theta^2) + (1+f)^2 - 2(1+f)f\rho\theta]\}.\end{aligned} \tag{11}$$

Propositions I, II, III, IV, V, derived in the subsequent sections, summarize our contribution to the literature. Net worth $X(t)$ equals capital less debt. Since capital/net worth less debt/net worth equals one, the propositions apply to the optimal ratio k^* of "capital"/net worth. Box 2 states the implications of **DP** equation (11) for the optimal debt/net worth, capital/net worth and consumption/net worth. *The economic interpretation is the subject of the subsequent sections.*

Proposition I. *The optimal debt/net worth f^* and capital/net worth k^* maximize a mean–variance function of expected return and risk.*

[9] In the models that use IBC, one must know the expected present value of future income over (say) an infinite horizon. In the model in Box 1, such a concept is unknowable.

Proposition II. *The optimal* f^* *or* $k^* = 1 + f^*$ *are independent of the optimal ratio of consumption/net worth and discount rate. This "separation theorem" is seen directly from an inspection of the* $\max_f\{.\}$ *term in Eq.* (11).

Proposition III. *When utility is logarithmic, the optimal debt/net worth* f^* *or capital/net worth* k^* *maximize the expected endogenous growth rate, for any constant consumption ratio.*

Proposition IV. *The optimal debt/net worth will only be positive if the expected return exceeds the expected real interest rate by an amount that depends upon the correlation of the growth and interest rate risks and their variances.*

Proposition V. *The optimal expected current account deficit/net worth is a function of the optimal debt/net worth. Permanent current account deficits/net worth are optimal if* f^* *and expected growth are positive.*

Box 2. Summary of optimal (*) controls

Debt/net worth

$$f^* = (b - r)/(1 - \gamma)\sigma^2 + \lambda(\rho\theta - 1) = (b - r)/(1 - \gamma)\sigma^2 + f(0). \tag{12}$$

Capital/net worth

$$[(K(t))/X(t)]^* = k^* = 1 + f^* \geqslant 0. \tag{13}$$

Consumption/net worth

$$c^* = C(t)/X(t) = A^{-1/(1-\gamma)}, \quad c^* = \delta, \quad \text{when} \quad \gamma = 0. \tag{14}$$

Symbols: *Net worth* $X(t) = K(t) - L(t)$; expected net return $= (b - r)$; total risk $= \sigma^2 = \mathrm{var}(b(t) - r(t)) = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) > 0$; $\theta = \sigma_1/\sigma_2 =$ standard deviation of interest rate/standard deviation of growth; $\rho =$ correlation between interest rate and growth; $\lambda = (\sigma_2^2/\sigma^2) = 1/(1 + \theta^2 - 2\rho\theta) > 0$. Intercept term $f(0) = \lambda(\rho\theta - 1)$.

In the model, the expected return on investment b in Eq. (3) is a constant: [10] there are no diminishing returns. Similarly, the expectation of the real interest rate r in Eq. (6) is constant. Assume that $b > r$. In the *conventional approach*, the optimal stock of capital is such that the expected return is equal to the interest rate. Since $b > r$, the country should increase its capital without limit. Insofar as the saving ratio is given, the debt should rise without limit.

The DP approach yields a different result. Eq. (12), graphed in Fig. 1 as curve U–S, relates the ratio f^* of the optimal debt/net worth to the expected net return

[10] It may be a slowly changing variable, as shown in the example in the concluding section.

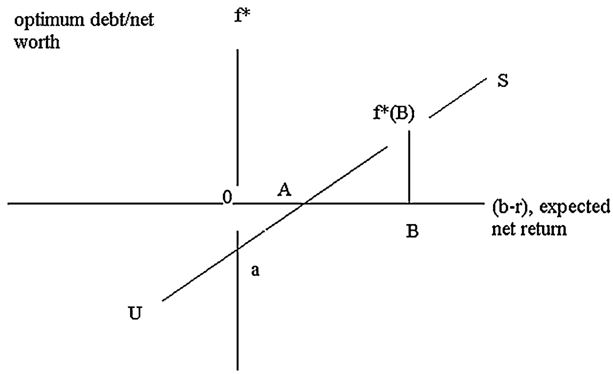

Fig. 1. Optimal debt/net worth f^*, expected net return $(b-r)$.

on investment $(b-r)$. The ratio of "capital"/net worth $k^* = 1 + f^*$, so that the graph can be used to determine either debt or capital relative to net worth. *In the section below, we explain in detail how this equation and the optimal growth equation can be understood in a mean–variance framework.*

The slope of the curve $1/(1-\gamma)\sigma^2$ is the reciprocal of "total risk" times risk aversion. Total risk σ^2 is the variance of the net return $= \mathrm{var}(b(t) - r(t)) = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) > 0$. The intercept $f(0) = \lambda(\rho\theta - 1)$ is the optimal ratio debt/net worth when the expected net return is zero. It can be positive, zero or negative. In the mathematical appendix, available upon request, we show that $f(0)$ is the ratio of debt/net worth that minimizes total risk.

Eq. (12)/Fig. 1 states that, as long as the expected net return is less than $A = \sigma_2^2(1-\gamma)(1-\rho\theta)$, the ratio of debt/net worth should be negative – the economy should be a creditor. As the expected net return rises above A, the economy should finance capital with debt. At expected net return B, the ratio of optimal debt/net worth is $f^*(B)$ and optimal capital/net worth is $1 + f^*(B)$, both are finite. A debt/net worth is excessive/non-optimal, insofar as it lies above a line such as U–S in Fig. 1.

4. A "mean–variance" (MV) interpretation

The Tobin mean–variance (M–V) analysis is the cornerstone of much of the work in the field of investment/portfolio allocation analysis. It is based upon a two period model of portfolio choice between "safe" and "risky" assets, and implies clear and operational results. Our model in Box 1 seems to be quite different. There is an infinite horizon, and there is risk on both the debt and on capital. A negative debt is a positive holding of financial assets. Growth is endogenous. We show how the DP equations in Box 2 can be related to the M–V analysis.

The optimal values of debt/net worth f^* or "capital"/net worth $k^* = 1 + f^*$ maximize the value function equation $V(X)$ in (1) subject to the law of motion of the state variable $X(t)$ net worth, Eq. (10a). In the M–V analysis, the object is to select a portfolio of risky and safe assets to maximize $V^* = M - (1-\gamma)R$, a linear combination of a mean M and $(1-\gamma)R$, risk R times risk aversion $(1-\gamma) > 0$. To relate the DP equation (12) for the optimal debt to the maximization of V^* in the M–V analysis we must have expressions for "mean" M and "risk" R, which are based upon the model in Box 1.

In general, for all positive risk aversion, the optimal consumption $C(t)$ will be a constant c^* times net worth $X(t)$. Therefore, the growth of consumption will equal the growth in net worth, Eq. (15).

$$(1/t)\ln[C(t)/C(0)] = (1/t)\ln[X(t)/X(0)] = \text{growth rate.} \tag{15}$$

Eq. (16), "expected growth", is derived from the solution of stochastic differential equation (10a). *If one starts from Eq.* (16), *we show how the dynamic programming results summarized in Box* 2 *can be given an interpretation in the traditional "mean–variance" portfolio choice model*. The proofs, based upon the stochastic calculus, are in the mathematical appendix, available upon request.

Expected growth of consumption and net worth:

$$\begin{aligned}(1/t)E[\ln C(t)/C(0)] &= (1/t)E[\ln X(t)/X(0)]\\ &= [(b-c)+(b-r)f]\\ &\quad -(\sigma_2^2/2)[f^2\theta^2+(1+f)^2-2f(1+f)\rho\theta]\\ &= M(f,c)-R(f).\end{aligned} \tag{16}$$

Divide Eq. (16) into two parts, which correspond to **M**ean and **R**isk, defined below in Eqs. (17) and (18) respectively. In the discussion here, the ratios f and c are assumed constant. The *mean return M is expected growth if there were no risks*. It is independent of the variances and covariances.

$$M = [(b-c)+(b-r)f]. \tag{17}$$

The mean return M depends upon: $(b-c)$ the expected return on investment less the ratio of consumption/net worth, plus the expected rate of return less the real interest rate $(b-r)$ times f the ratio of debt/net worth.

The variance of the growth rate $\mathrm{var}(1/t)\ln[X(t)/X(0)]$ is Eq. (18a), which is independent of the consumption ratio and depends upon one control variable, the debt/net worth.

Variance of consumption and growth:

$$\begin{aligned}(1/t)\mathrm{var}[\ln C(t)/C(0)] &= (1/t)\mathrm{var}[\ln X(t)/X(0)]\\ &= [f^2\sigma_1^2+(1+f)^2\sigma_2^2-2f(1+f)\rho\sigma_1\sigma_2]\\ &= \sigma_2^2[f^2\theta^2+(1+f)^2-2f(1+f)\rho\theta].\end{aligned} \tag{18a}$$

Define *Risk R*, Eq. (18), as equal to one half of the variance of growth. *Risk R* only contains variances, covariances and debt/net worth. The variance of the return is σ_2^2, the variance of the interest rate is σ_1^2, the ratio $\theta = \sigma_1/\sigma_2$ and ρ is the correlation between the disturbances.

$$R = (\sigma_2^2/2)[f^2\theta^2 + (1+f)^2 - 2f(1+f)\rho\theta]. \tag{18}$$

Define expected M–V utility as V^* in Eq. (19) equal to the **M**ean less the product of risk aversion $(1-\gamma) > 0$ and **R**isk.

$$V^*(f,c) = M(f,c) - (1-\gamma)R(f). \tag{19}$$

There is a correspondence between the DP solution, based upon stochastic optimal control equation (11), and the M–V approach equation (19), because DP equation (11) can be written as Eq. (20) using the definitions for "mean" M and risk "R" above. *Eq.* (20) *shows that the maximization with respect to the optimal debt/net worth is the same in either approach*. Recall that a negative debt is a positive financial asset position.

$$\begin{aligned}\delta/\gamma &= \max_{c,f}\{(1/\gamma)c^\gamma/A + M(f,c) - (1-\gamma)R(f)\}\\ &= \max\{(1/\gamma)c^\gamma/A + V^*(f,c)\}.\end{aligned} \tag{20}$$

A graphic discussion of the correspondence between the two approaches, for the optimum debt/net worth, is the subject of the next section. The economic analysis proceeds on the basis of the derived graph.

5. Optimal ratio of debt/net worth: Mean–variance and a generalization of the Merton solution

A M–V interpretation of Eq. (12) for the optimal debt/net worth is done graphically in Fig. 2, where we select a debt/net worth ratio that maximizes the "mean–variance expected utility" $V^* = M(f,c) - (1-\gamma)R(f)$, Eq. (19).

The mean $M(f,c)$ in (17) is a linear function of f the debt/net worth. The slope of the mean function is $\mathrm{d}M/\mathrm{d}f = (b-r)$, the expected return less the expected interest

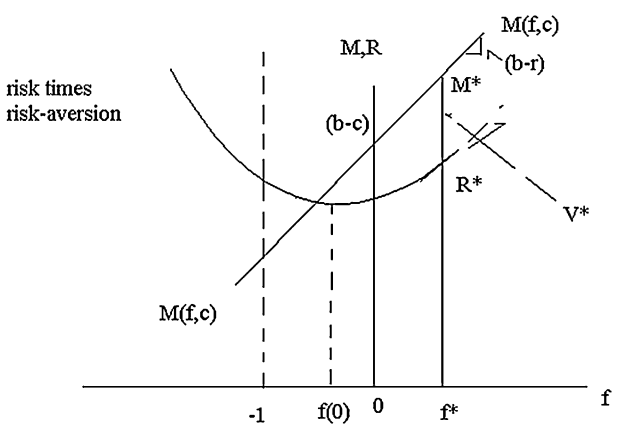

Fig. 2. Mean–variance interpretation of DP equation.

rate, is independent of the debt and consumption. There are no diminishing returns to investment. Intercept $(b-c)$ is the expected return less the consumption ratio. Variations in the consumption ratio only affect the intercept and not the slope of the mean function.

Risk $R(f)$ in (18) is a quadratic function of the debt/net worth, which is independent of consumption and the net return. Total risk $R(f)$ is not the same as the variance of the return on investment σ_2^2. Borrowing to finance real investment involves a risky return and a risky interest rate liability. The two risks may be correlated positively or negatively, or may be independent of each other. The uncertainty concerns the variance of the net return.

Quadratic risk function $R(f)$ reaches a minimum at $f(0)$ in Fig. 2 and rises as the net debt/net worth deviates from $f(0)$. The minimum risk ratio of debt/net worth at $f(0) = (\rho\theta - 1)/(1 + \theta^2 - 2\rho\theta)$, is the intercept term in Eq. (12) and Fig. 1. To minimize risk, the country should be a debtor (creditor) if quantity $(\rho\theta - 1)$ is positive (negative).

The mean–variance interpretation of the DP Eq. (11) is that the optimal ratio f^* of debt/net worth in Fig. 2 maximizes expected M–V utility V^* equal to the difference between mean return and risk times risk aversion. $f^* \in \mathrm{argmax}[V^* = M(f,c) - (1-\gamma)R(f)]$.

This *optimal ratio is precisely the* f^* *in Eq.* (12), *derived from the DP solution of the stochastic optimal control/infinite horizon model.*

Since the optimal ratio of "capital"/net worth is $k^* = 1 + f^*$, we could have used the maximization with respect to k instead of with the debt/net worth ratio. Our approach is a generalization of the Merton model to an open economy with two types of risk. In Merton's model the investor has wealth $X(t)$ which he divides between a risky asset and a safe asset. The price of the risky asset follows a Brownian motion process similar to our Eq. (3), and there is no interest rate risk $\sigma_1 = 0$. The well-known Merton equation (1990: 111) for the ratio of risky assets/net worth k^* is a special case of our equation (12). Since there is no interest rate risk: $\theta = 0$, $\rho = 0$, $\lambda = 1$. The resulting ratio of risky assets/net worth, $k^* = 1 + f^* = (b - r)/(1-\gamma)\sigma_2^2$, is the well-known Merton solution. We have shown that our DP approach generalizes the Merton model. Since we both use dynamic programming, we both obtain results very different from the open economy models, which use either the intertemporal budget constraint or the Maximum Principle.

6. Expected growth and MV utility

Expected growth g, the M–V expected utility V^* and the DP equation are intimately related, as can be seen from Eqs. (16), (19) and (20). The implications of the DP approach for optimal endogenous growth are quite different from that found in the literature.

$$g = (1/t)E[\ln X(t)/X(0)] = M(f,c) - R(f). \tag{16}$$

$$V^*(f,c) = [M(f,c) - (1-\gamma)R(f)]. \tag{19}$$

$$\begin{aligned}\delta/\gamma &= \max_{c,f}\{(1/\gamma)c^{\gamma}/A + V^*(f,c)\}\\ &= \max_{c,f}\{(1/\gamma)c^{\gamma}/A + M(f,c) - (1-\gamma)R(f)\}.\end{aligned} \tag{20}$$

First: the optimal expected growth rate depends upon the optimal debt/net worth f^* and the optimal consumption/net worth c^*. The optimal ratio f^* of debt/net worth is independent of the discount rate, which effectively determines the planning horizon. A myopic planner, with large delta, has the same optimal f as one with a longer planning horizon.

Second: the optimal ratio f^* of debt/net worth is independent of the consumption ratio c. Variations in the consumption ratio c change the intercept, but not the slope, of the line $M(f,c)$, and do not affect the curve for risk $R(f)$ in Fig. 2.

Third: when the utility function is logarithmic, $\gamma = 0$, then expected growth g in Eq. (16) is the same as expected M–V utility V^* in (19). The difference between the straight line $M(f,c)$ and the risk curve $R(f)$ in Fig. 2 is expected M–V utility equal to expected growth g. For any given consumption ratio, the optimal debt/net worth ratio f^* maximizes the expected growth rate.

Fourth: the optimal consumption/net worth ratio balances the two terms in the maximization in (20). A decline in the consumption ratio increase M–V utility and the expected growth rate V^* in Eq. (16) by shifting the mean line $M(f,c)$ upwards. However, from DP equation (20), the optimal consumption ratio must also take into account current consumption – the first term – and not just M–V expected utility V^*.

Fifth: As proved in the mathematical appendix, available upon request, optimal consumption/net worth c^* is a constant. If the utility function is logarithmic, then the optimal ratio of consumption/net worth [11] is equal to the discount rate δ in Eq. (21).

$$c^* = C(t)/X(t) = \delta. \tag{21}$$

Sixth: In the logarithmic case, the optimal endogenous growth g^* is Eq. (22). It is the difference between the $M(f,c)$ line and $R(f)$ curve, evaluated at c^*, f^* from Eqs. (21) and (12) respectively.

$$g^* = M(f^*,c^*) - R(f^*) = M(f^*,\delta) - R(f^*). \tag{22}$$

This equation for optimal growth differs from the growth rate derived from the Solow model. There are no diminishing returns to capital, the consumption ratio is optimal, the ratio of debt to net worth is optimal and risk is explicitly taken into account. The DP approach ties together optimal debt/net worth, optimal endogenous growth and mean–variance analysis.

[11] Fleming (2001) and Merton (1990: 111) have derived the optimal consumption ratio in the general case where $1 > \gamma$.

7. Optimum current account

In the context of international finance, it is frequently argued that continued current account deficits are unsustainable and increase the probability of a crisis. On the basis of our analysis based upon stochastic optimal control we answer the following questions: When optimal policies are followed, what is the expected current account? Can it be optimal that the rich countries in the world be debtors? What is a sustainable current account deficit? We contrast the implications of our stochastic optimal control/dynamic programming approach with that implied by the IBC literature. The main proposition implied by the IBC literature is that the optimal current account is equal to the difference between current national income and from its "permanent" level less the deviation of government consumption from its "permanent" level. [12] The "permanent" level of national income is the annuity value of the expected present value of national income.

Despite its great popularity in the theoretical literature, the IBC has not been used in *empirical or policy oriented* work to evaluate whether the current account deficit is sustainable or optimal. [13] The reason for this disparity between "theory" and "empirical/policy" is that IBC literature is not operational. The Intertemporal Budget Constraint propositions are unenforceable. No one can know with reasonable confidence what is the expected present value of future GDP. When current account deficits are incurred, no one can say with any confidence that they just reflect consumption smoothing and that they will be reversed in the future, so that there will not be a debt crisis. There is no objective measure of what is an unsustainable situation. There is no feedback control to correct errors. As more information is obtained about future real income, how should previous errors – excessive trade deficits – be corrected?

The results in Box 2/Figs. 1 and 2 show why the stochastic optimal control/dynamic programming approach arrives at propositions very different from the IBC propositions above. In our DP analysis, permanent current account deficits will be optimal if the optimal debt/net worth f^* and growth are positive. Then the debt should grow at the same rate as net worth. Since the current account deficit is the change in the debt, it is optimal to have continuing current account deficits. The derivation of the optimal current account deficit is as follows. Since the ratio $f^* = L(t)/X(t)$ is the ratio of optimal debt/net worth, the optimal current account deficit/net worth is [14] (23) where $\mathrm{d}L^*(t)$ is the change in the debt.

$$\mathrm{d}L^*(t) = f^*\,\mathrm{d}X(t). \tag{23}$$

[12] See Gandolfo: 305.

[13] It is not always clear whether it is claimed that the observed market behavior is the optimal behavior described by the IBC literature or whether that concept of optimality is just a benchmark. Here, we take the weaker interpretation that the IBC provides a benchmark. The IBC literature is discussed fully in Gandolfo (2001, ch. 18–19).

[14] The optimal debt $L^*(t) = f^*X(t) = L(X(t))$, where $L_x = f^*$ and $L_{xx} = 0$. Therefore the change $\mathrm{d}L(t) = L_x\,\mathrm{d}X(t) + (L_{xx}/2)(\mathrm{d}X(t))^2 = f^*\,\mathrm{d}X(t)$.

The optimal ratio f^* is a control variable in this model and is constant. The actual change in net worth $\mathrm{d}X(t)$ in Eq. (10a) has two components: a mean $M(f,c) = [(b-c)+(b-r)f]X(t)$ and a stochastic part containing the two Brownian motion terms with zero expectations. The actual change in net worth will jump around due to the Brownian motion terms. The expectation of the change in the debt $E(\mathrm{d}L(t))$/net worth $X(t)$, denoted by $Z(t)$, is Eq. (24). This is the optimal expected current account deficit/net worth.

$$\begin{aligned} Z(f^*,c^*) &= E(\mathrm{d}L(t))/X(t) = f^*E[\mathrm{d}X(t)]/X(t) = f^*M(f^*,c^*) \\ &= f^*[(b-c^*)+(b-r)f^*]. \end{aligned} \tag{24}$$

The expected optimal current account deficit/net worth is the product of the optimal debt/net worth ratio f^* and $M(f^*,c)$ which is the "mean" in the M–V analysis, Eq. (19), or a point on the straight line in Fig. 2. When the optimal debt $f^* > 0$, there will be permanent current account deficits/net worth.

We may sum up the differences between the DP approach and the IBC literature concerning the optimal current account as follows. (a) In the IBC literature, the optimum current account at any time depends upon the difference between current and "permanent income". The IBC is unknowable and cannot be enforced at any time. (b) In the DP approach, a permanent debtor or creditor position may be optimal. It all depends upon whether the optimal debt/net worth f^* in Eq. (12), and expected growth, are positive. The value of $f^* > 0$ if the mean net return $(b-r) > \lambda(1-\rho\theta)/(1-\gamma)\sigma^2$. Insofar as the optimal debt/net worth is a positive constant, such as point B in Fig. 1, then permanent current account deficits are required to maintain the ratio constant. (c) The current account deficit/net worth should be stationary if the expected net return $(b-r)$ is stationary.

8. Conclusion: Examples

Our stochastic optimal control approach has been applied in two papers, Stein and Paladino (2001) and Stein (2003). In the first paper, we evaluated the country default risk in emerging markets. In this case, the debt is short-term sovereign debt that must be repaid at maturity. This paper is an application of the stochastic optimal controls models of Fleming and Stein (2001) in discrete time and with a finite horizon. We provide benchmarks to evaluate an *optimal* debt and a *maximal* debt (*debt-max*), when risk is explicitly considered. Defaults are likely to occur when the servicing of the debt requires a decline in consumption. When the debt exceeds *debt-max*, the expected consumption will decline. Therefore, when the actual debt exceeds *debt-max*, then the economy will default when a "bad shock" occurs. We consider two sets of high-risk countries during the period 1978–99: a subset of 21 countries that defaulted on the debt, and another set of 13 countries that did not default. Default is a situation where the firms or government of a country reschedule the interest/principal payments on the external debt. We thereby explain how our analysis can anticipate default risk, and add another dimension to the literature of early warning signals of default/credit risk.

Our current paper involves an infinite horizon and continuous time, and "debt" includes equity and long-term bonds. There have not been debt crises in the major countries in recent years. In principle, our current paper is applicable to a country, a group of countries such as the European Monetary Union, a region within a country or a large sector within a country. There was a severe agricultural debt crisis in the US during the first half of the 1980s. For this reason, Stein (2003) applies the model in present paper to explain the US agricultural debt crisis. There are reliable data for the US agricultural sector, which relates directly to the theoretical variables. There is just one currency and the values of income, debt, interest expense, capital and equity are reported regularly. Since there are empirical measures of the *objective* components of the optimal debt f^*, we have chosen to cite this example of the advantages of our DP approach. [15]

Besides the parameters in our model to be estimated from data, there are two model design parameters to be freely chosen. These subjective variables are discount rate/time preference δ, and the exponent γ of the HARA function. Delta effectively determines the planning horizon and $(1-\gamma)$ is a measure of risk aversion. The optimal debt/equity ratio f turns out not to depend on time preference, and thus is not affected by the planning horizon. A myopic planner, with large delta, has the same optimal f as one with a longer planning horizon.

Historical data show that over a many year horizon, the mean productivity b and mean interest rate r in the model vary considerably depending on the economic environment. Our technique is to replace them with short term (five year) moving averages of $b(t)$ and $r(t)$ and then use these averaged values over shorter time scales as the true means b and r in the formula for optimal f. The variances and correlations are also time varying. The optimal f is then also time varying, primarily because the $(b-r)/\sigma^2$ varies; but the intercept $f(0)$ has not changed drastically over the periods. Therefore, the optimal f^* follows $(b-r)$ as five-year moving averages of past net returns.

The farm debt crisis occurred during the period 1980–88, the shaded area in Fig. 3, when defaults and delinquency ratios rose drastically. We plot in normalized form, [16] to facilitate comparisons, the actual debt/equity ratio DEBTEQUITY, the mean net return $(b-r)_5 = \text{RETVAINTD5}$ based upon past information and the ratio of interest payments/value added INTVA $= rL/Y$.

On the basis of this figure we see that during the period 1975–85 movements in the actual debt/equity ratio DEBTEQUITY fail to accord with those in the mean net return $(b-r)_5 = \text{RETVAINTD5}$. At any time $(b-r)_5$ is known from past data. From 1975–83 the five-year MA net return was declining; and from 1979–84 the actual net return was negative. The actual numbers are as follows.

[15] The full details are in Stein (2003), which is available on request.

[16] A variable X is X' in normalized form: $X'(t) = (X(t) - \text{mean})/\text{standard deviation}$. Thus the units of X' are in standard deviations.

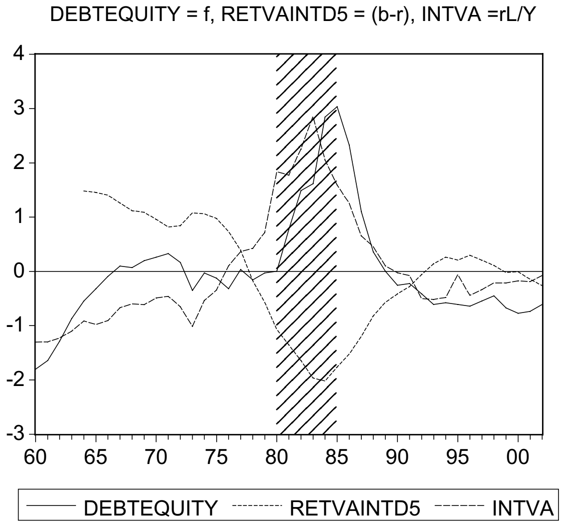

Fig. 3. Debt/equity ratio $f(t)$, expected net return $(b-r)$ as five-year MA, interest expense/value added $= rL/Y$.

	Debt/equity	Mean net return $(b-r)$	Interest expense/ value added
1975	20%	4.55%	11.8%
1985	29.8	−1.87	22.48

The optimal debt/equity ratio should have been declining. In terms of Fig. 1, when the mean net return declines from B to A, the optimal ratio declines from $f^*(B)$ to zero. Our analysis correctly predicts a debt crisis in the shaded region. On the other hand, our analysis correctly predicts tranquil periods pre-1979 and post-1990.

We have cited two examples of the usefulness of our technique: one for the emerging market countries and the other for the debt crisis in US agriculture.

9. Uncited references

The following references are cited in the mathematical appendix (available upon request): Fleming (1999); Fleming and Soner (1992) and Fleming and Rishel (1975).

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